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Multiple solutions for eigenvalue problems involving an indefinite potential and with $(p_1(x), p_2(x))$ balanced growth

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Abstract

In this paper we are concerned with the study of the spectrum for a class of eigenvalue problems driven by two non-homogeneous differential operators with different variable growth and an indefinite potential in the following form

$$-\mathrm{div}\Big[\mathcal{H}(x,|\nabla u|)\nabla u + \mathcal{I}(x,|\nabla u|)\nabla u\Big] + V(x)|u|^{m(x)-2}u = \\ = \lambda\left(|u|^{q_1(x)-2} + |u|^{q_2(x)-2}\right)u \text{ in } \Omega,$$

which is subjected to Dirichlet boundary condition. The proofs rely on variational arguments and they consist in finding two Rayleigh-type quotients, which lead us to an unbounded continuous spectrum on one side, and the nonexistence of the eigenvalues on the other.

1 Introduction

The study of variational problems with nonstandard growth conditions has been developed extensively over the last years. The $p(x)$-growth conditions can be regarded as a key factor in the modeling of some fluids which have different inhomogeneities, for instance the electrorheological fluids. This leaded to the necessity of studying the variable exponent Lebesgue and Sobolev spaces,

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$L^{p(x)}$ and $W^{1,p(x)}$, where p is a real valued function. The variable exponent Lebesgue and Sobolev spaces also play an important role in the study of the thermorheological fluids, the development of the robotics, aircraft and airspace and the image restoration.

We are interested in the study of a class of stationary problems, which are characterized by the fact that the associated energy density changes its ellipticity and growth properties according to the point.

This new type of non-homogeneous differential operators has been introduced by I. H. Kim and Y.H. Kim in [12], and it helps us to understand the nonlinear problems with possible lack of uniform convexity.

In this paper we extend the results obtained by M. Mihăilescu and V. Rădulescu in [15] in the framework of the new operators introduced by I. H. Kim and Y. H. Kim. We study the presence of two operators with variable growth and the influence of an indefinite sign-changing potential on their spectral properties.

We consider the following nonlinear eigenvalue problem:

$$(P)\begin{cases} -\operatorname{div}\Big[\mathcal{H}(x,|\nabla u|)\nabla u+\mathcal{I}(x,|\nabla u|)\nabla u\Big]+V(x)|u|^{m(x)-2}u= & \\ \qquad\qquad = \lambda\left(|u|^{q_1(x)-2}+|u|^{q_2(x)-2}\right)u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $V(x)$ is an indefinite sign-changing potential and $\lambda \in \mathbb{R}$ is a real parameter. We note that similar results were obtained by S. Baraket, S. Chebbi, N. Chorfi and V. Rădulescu in [2], under the basic assumption that $V \equiv 0$ and the right-hand side of the problem has only a $q(x)$-growth rate.

The study of the $(p_1(x), p_2(x))$-growth rate problems was motivated by the fact that we may need to model a composite that changes its hardening exponent according to the point. This kind of problems was also studied by P. Marcellini [13, 14].

This type of problems was also studied by G. Mingione *et al.* [3, 6, 7] in the framework of two different materials with power hardening exponents $p_1(x)$ respectively $p_2(x)$, and a coefficient $a(x)$ which dictates the geometry of a composite of the two materials as it follows:

$$u \mapsto \int_\Omega \Big[|\nabla u|^{p_1(x)} + a(x)|\nabla u|^{p_2(x)} \log(e+|x|)\Big]\,dx.$$

In our research we obtain three important results which are: the revealing of the infimum eigenvalue associated to our problem, the nonexistence of the eigenvalues for quantities smaller than one Rayleigh-type quotient and a final result which points out the concentration of the spectrum associated to the non-homogeneous operator. For more details about spectral properties of differential operators we may refer to [12, 16] and to Chapter 3 of [19].

More contributions to the study of the eigenvalue nonlinear elliptic equations in an anisotropic framework were also added by K. Ben Ali, A. Ghanmi, K. Kefi [1], M. Cavalcanti, V. Domingos Cavalcanti, I. Lasiecka, C. Webler [4], M. Cencelj, D. Repovš, Z. Virk [5], Y. Fu, Y. Shan [10], K. Kefi, V. Rădulescu [11], D. Repovš [20] and I. Stăncuţ, I. Stîrcu [21].

2 The functional framework

With the emergence of nonlinear problems in applied sciences, standard Lebesgue and Sobolev spaces demonstrated their limitations in the applications. The class of nonlinear problems with variable exponent growth reflects a new kind of physical phenomena.

In order to deal with the problem (P) we need some theory about the generalized Lebesgue-Sobolev spaces. In what follows we will give a simple description and we will recall their main properties. These results are described in the following books: J. Musielak [17], L. Diening, P. Hästö, P. Harjulehto, M. Ružička [8], V. Rădulescu and D. Repovš [19]. We also refer to the survey paper by V. Rădulescu [18].

Let Ω be a bounded domain in $\mathbb{R}^N$.

For a measurable function $p : \overline{\Omega} \to \mathbb{R}$ we define:

$$p^+ = \sup_{x\in\Omega} p(x) \qquad \text{and} \qquad p^- = \inf_{x\in\Omega} p(x).$$

Set:

$$C_+(\Omega) = \left\{p \in C(\overline{\Omega}) : \ p(x) > 1, \text{ for all } x \in \overline{\Omega}\right\}.$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined

$$L^{p(x)}(\Omega) = \left\{u; \ \ u : \Omega \to \mathbb{R} \text{ a measurable function} : \int_\Omega |u|^{p(x)} dx < \infty\right\},$$

and with the norm:

$$|u|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\},$$

$L^{p(x)}(\Omega)$ becomes a Banach space whose dual is the space $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Remark 2.1. If $1 < p(x) < \infty$, $L^{p(x)}(\Omega)$ is reflexive Banach space. Moreover, if p is measurable and bounded, then $L^{p(x)}(\Omega)$ is also separable.

Remark 2.2. If $0 < |\Omega| < \infty$ and $h(x)$, $r(x)$ with $h(x) < r(x)$ almost everywhere in Ω, are two variable exponents then the following continuous embedding holds

$$L^{r(x)}(\Omega) \hookrightarrow L^{h(x)}(\Omega).$$

Let $L^{p'(x)}(\Omega)$ denotes the dual space of $L^{p(x)}(\Omega)$. For all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the following Hölder type inequality holds:

$$\left| \int_\Omega uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)} \leq 2|u|_{p(x)} |v|_{p'(x)}. \tag{1}$$

A key role in the studies which imply the variable exponent Lebesgue spaces is played by the modular of $L^{p(x)}(\Omega)$, which is $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ and is defined by

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

Remark 2.3. If $p(x) \not\equiv constant$ in Ω, for u, $(u_n) \in L^{p(x)}(\Omega)$, the following relations hold true:

$$|u|_{p(x)} < 1 \Rightarrow \; |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}, \tag{2}$$

$$|u|_{p(x)} > 1 \Rightarrow \; |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}, \tag{3}$$

$$|u|_{p(x)} = 1 \Rightarrow \; \rho_{p(x)}(u) = 1, \tag{4}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \; \rho_{p(x)}(u_n - u) \to 0. \tag{5}$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

On $W^{1,p(x)}(\Omega)$ we may consider the following equivalent norms:

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

and

$$\|u\| = \inf \left\{ \mu : \int_\Omega \left(\left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} + \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) dx \leq 1 \right\}.$$

We define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\| \cdot \|_{p(x)}$ or

$$W_0^{1,p(x)}(\Omega) = \left\{ u; u|_{\partial\Omega} = 0, u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

Taking account of [12] for $p \in C_+(\overline{\Omega})$ we have the $p(\cdot)$-*Poincaré type inequality*

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}, \tag{6}$$

where $C > 0$ is a constant which depends on p and Ω.

For $\Omega \subset \mathbb{R}^N$ a bounded domain and p a global log-Hölder continuous function, on $W_0^{1,p(x)}(\Omega)$ we can work with the norm $|\nabla u|_{p(x)}$ equivalent with $\|u\|_{p(x)}$.

Remark 2.4. If $p, q : \Omega \to (1, \infty)$ are Lipschitz continuous, $p^+ < N$ and $p(x) \leq q(x) \leq p^*(x)$, for any $x \in \Omega$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$, the embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is compact and continuous.

Remark 2.5. If $0 < |\Omega| < \infty$, and $p_2(x) < p_1(x)$ in Ω, then there holds the following continuous embedding

$$W_0^{1,p_1(x)}(\Omega) \hookrightarrow W_0^{1,p_2(x)}(\Omega).$$

3 Basic hypotheses

We will study the problem

$$(P) \begin{cases} -\mathrm{div}\Big[\mathcal{H}(x,|\nabla u|)\nabla u + \mathcal{I}(x,|\nabla u|)\nabla u\Big] + V(x)|u|^{m(x)-2}u = & \\ \qquad\qquad = \lambda\left(|u|^{q_1(x)-2} + |u|^{q_2(x)-2}\right)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

In order to state more precisely our results we have that:

(HI_1) $\mathcal{H}, \mathcal{I} : \Omega \times [0, \infty) \to [0, \infty)$ satisfy the following assumptions:

$\to \mathcal{H}(\cdot, t)$ and $\mathcal{I}(\cdot, t)$ are measurable on Ω for all $t \geq 0$;

$\to \mathcal{H}(x, \cdot)$ and $\mathcal{I}(x, \cdot)$ are locally absolutely continuous on $[0, \infty)$ for almost all $x \in \Omega$.

(HI_2) There exist $\alpha_1 \in L^{p_1'(x)}(\Omega)$ and $\alpha_2 \in L^{p_2'(x)}(\Omega)$ and some positive constants $\beta_1, \beta_2 > 0$ such that:

$\to |\mathcal{H}(x, |v|)v| \leq \alpha_1(x) + \beta_1|v|^{p_1(x)-1}$;

$\to |\mathcal{I}(x, |v|)v| \leq \alpha_2(x) + \beta_2|v|^{p_2(x)-1}$;

for almost all $x \in \Omega$ and for all $v \in \mathbb{R}^N$.

(HI_3) There is a positive constant c such that the following hypotheses hold for almost all $x \in \Omega$:

$\to \mathcal{H}(x, t) \geq ct^{p_1(x)-2}$, $\ t\dfrac{\partial \mathcal{H}}{\partial t}(x, t) + \mathcal{H}(x, t) \geq ct^{p_1(x)-2}$;

$\to \mathcal{I}(x,t) \geq ct^{p_2(x)-2}$, $t\dfrac{\partial \mathcal{I}}{\partial t}(x,t) + \mathcal{I}(x,t) \geq ct^{p_2(x)-2}$;
for all $t > 0$.

Let us now impose some conditions over our variable exponents. Suppose that we have $p_1, p_2, q_1,\, q_2,\, m : \overline{\Omega} \to (1, \infty)$, $p_1, p_2, q_1, q_2, m \in C_+(\overline{\Omega})$,

$$\begin{aligned} p_2^+ < q_2^- \leq q_2(x) \leq q_2^+ \leq m^- \leq m(x) \leq m^+ \leq q_1^- \leq q_1(x) \leq q_1^+ < \\ < p_1^- \leq p_1(x) \leq p_1^+ \end{aligned} \tag{7}$$

and

$$q_1^+ < p_2^*(x) := \begin{cases} \frac{Np_2(x)}{N-p_2(x)}, & \text{if } p_2(x) < N; \\ +\infty, & \text{if } p_2(x) \geq N. \end{cases} \tag{8}$$

The function $V : \Omega \to \mathbb{R}$ is an indefinite sign-changing potential and it satisfies:
(V) $V \in L^{r(x)}(\Omega)$ with $r \in C_+(\overline{\Omega})$ and $r(x) > \frac{N}{m^-}$, for all $x \in \overline{\Omega}$.

We remark that for the more strictly case when $\mathcal{H}(x,t) = t^{p_1(x)-2}$ and $\mathcal{I}(x,t) = t^{p_2(x)-2}$ our differential operators become the $p(x)$-Laplace operator $(\Delta_{p(x)}(u) := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u))$. We may also observe that if $\mathcal{H}(x,t)$ and $\mathcal{I}(x,t)$ are of type $\mathcal{H}(x,t) = (1+|t|^2)^{\frac{p(x)-2}{2}}$ and $\mathcal{I}(x,t) = (1+|t|^2)^{\frac{p(x)-2}{2}}$, our operators become the generalized mean curvature operator:

$$\operatorname{div}\left[(1+|\nabla u|^2)^{\frac{p(x)-2}{2}}\nabla u\right],$$

which drive us to the capillary surface operator

$$\operatorname{div}\left[\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right].$$

These types of general differential operators are also studied by D. Repovš in [20].

Our hypotheses (7) and (8) dictates the fact that we should search our weak solutions in the space $W_0^{1,p_1(x)}(\Omega)$.

Definition 3.1. We say that $u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}$ is a weak solution of the problem (P) if

$$\int_\Omega \mathcal{H}(x,|\nabla u|)\nabla u \nabla\varphi + \mathcal{I}(x,|\nabla u|)\nabla u\nabla\varphi dx + \int_\Omega V(x)|u|^{m(x)-2}u\varphi dx =$$

$$= \lambda \int_\Omega |u|^{q_1(x)-2}u\varphi + |u|^{q_2(x)-2}u\varphi dx,$$

for all $\varphi \in W_0^{1,p_1(x)}(\Omega)$.

For now on we set $W := W_0^{1,p_1(x)}(\Omega)$ and $\|\cdot\| := \|\cdot\|_{p_1(x)}$.

We note that if λ is an eigenvalue of (P), then the corresponding eigenfunction $u \in W \setminus \{0\}$ is a weak solution for the problem (P).

Using the hypotheses $(HI_1) - (HI_3)$ we define:

$$E_0(x,t) := \int_0^t \mathcal{H}(x,\xi) \cdot \xi d\xi + \int_0^t \mathcal{I}(x,\xi) \cdot \xi d\xi.$$

A key role in finding our weak solution is played by the following assumption:

(HI_4) $0 \leq \mathcal{H}(x,|t|)|t|^2 + \mathcal{I}(x,|t|)|t|^2 \leq p_1^+ E_0(x,|t|)$,
holds for all $x \in \overline{\Omega}$ and for all $t \in \mathbb{R}^N$.

4 On the infimum eigenvalue

Now, for each potential $V \in L^{r(x)}(\Omega)$ we consider the following Rayleigh-type quotients:

$$\lambda_1(V) := \inf_{u \in W\setminus\{0\}} \frac{\displaystyle\int_\Omega E_0(x,|\nabla u|)dx + \int_\Omega \frac{V(x)}{m(x)}|u|^{m(x)}dx}{\displaystyle\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx}$$

and

$$\lambda_0(V) := \inf_{u \in W\setminus\{0\}} \frac{\displaystyle\int_\Omega \left[\mathcal{H}(x,|\nabla u|) + \mathcal{I}(x,|\nabla u|)\right]|\nabla u|^2 dx + \int_\Omega V(x)|u|^{m(x)}dx}{\displaystyle\int_\Omega |u|^{q_1(x)}dx + \int_\Omega |u|^{q_2(x)}dx}.$$

Due to the quantities $\lambda_1(V)$ and $\lambda_0(V)$ we define the following functionals $E, \mathcal{E}_V, \mathcal{F} : W \to \mathbb{R}$, by:

$$E(u) := \int_\Omega E_0(x,|\nabla u|)dx,$$

$$\mathcal{E}_V(u) := E(u) + \int_\Omega \frac{V(x)}{m(x)}|u|^{m(x)}dx,$$

$$\mathcal{F}(u) := \int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx.$$

Using Lemma 3.2 from [12] and arguments from Chapter 3 of [19], we have that $\mathcal{E}_V(u)$ and $\mathcal{F}(u) \in C^1(W,\mathbb{R})$ and for all $x \in \mathbb{R}^N$ and $v \in W$ we have:

$$\langle \mathcal{E}_V'(u), v\rangle = \int_\Omega \left[\mathcal{H}(x,|\nabla u|) + \mathcal{I}(x,|\nabla u|)\right] \nabla u \nabla v dx + \int_\Omega V(x)|u|^{m(x)-2}uv dx$$

and

$$\langle \mathcal{F}'(u), v\rangle = \int_\Omega \left(|u|^{q_1(x)-2} + |u|^{q_2(x)-2}\right) uvdx.$$

We proceed now to reveal our first result.

Theorem 4.1. *The quantity $\lambda_1(V)$ is an eigenvalue of problem (P) and the corresponding eigenfunction u is a weak solution for our problem.*

Before we proceed to show our result, we will firstly prove some important properties.

Proposition 4.2. *Assuming that conditions (7), (8) and (V) hold, then for every constant $\varepsilon > 0$, we may find a constant $C_\varepsilon > 0$ such that:*

$$\left|\int_\Omega \frac{V(x)}{m(x)}|u|^{m(x)}dx\right| \leq \varepsilon \int_\Omega \left(|\nabla u|^{m^-} + |\nabla u|^{m^+}\right) dx$$
$$+ \quad C_\varepsilon |V|_{r(x)} \int_\Omega \left(|u|^{m^-} + |u|^{m^+}\right) dx,$$

for all $u \in W$.

Proof. Since $r(x) \geq r^-$ on $\overline{\Omega}$ we have that $L^{r(x)}(\Omega) \subset L^{r^-}(\Omega)$. Using assumption (V) we also have that $r(x) > \frac{N}{m^-}$, for all $x \in \overline{\Omega}$ and it follows that $r^- > \frac{N}{m^-}$ and $V \in L^{r^-}(\Omega)$.

We fix $\varepsilon > 0$. We search a constant $C_{V,\varepsilon}$ such that for all $u \in W_0^{1,m^-}(\Omega)$ we have that

$$\int_\Omega |V(x)| \cdot |u|^{m^-} dx \leq \varepsilon \int_\Omega |\nabla u|^{m^-} dx + C_{V,\varepsilon}|V|_{r^-} \int_\Omega |u|^{m^-} dx. \tag{9}$$

Firstly we aim to reveal that for each $z \in (1, {m^-}^*)$, where ${m^-}^* := \frac{Nm^-}{N-m^-}$, we may find a constant $C'_{V,\varepsilon} > 0$ such that:

$$|\xi|_z < \varepsilon\, |\,|\nabla \xi|\,|_{m^-} + C'_{V,\varepsilon}|\xi|_{m^-}, \ \ \forall\, \xi \in W_0^{1,m^-}(\Omega). \tag{10}$$

Arguing by contradiction we suppose that (10) does not hold for every $\varepsilon > 0$.

Then we may find a sequence $(\xi_n) \subset W_0^{1,m^-}(\Omega)$ with $|\xi_n|_z = 1$ and a strictly positive constant ε_0 such that

$$\varepsilon_0\, |\,|\nabla \xi_n|\,|_{m^-} + n|\xi_n|_{m^-} < 1, \ \ \forall\, n.$$

It is trivial to say that (ξ_n) is bounded in $W_0^{1,m^-}(\Omega)$ and $|\xi_n|_{m^-} \to 0$. Let $\xi \in W_0^{1,m^-}(\Omega)$ be the weak limit of ξ_n (passing eventually to a subsequence).

We can actually note that $\xi = 0$. Since $z \in (1, m^{-^*})$ we have the following compact embedding

$$W_0^{1,m^-}(\Omega) \hookrightarrow L^z(\Omega)$$

which implies

$$\xi_n \to 0 \quad \text{ in } L^z(\Omega).$$

But $|\xi_n|_z = 1$ which yields that $|\xi|_z = 1$, fact that is leading us to a contradiction. Thus we have proved that relation (10) holds.

Since $r^- > \frac{N}{m^-}$, a straightforward computation show us that $m^- \cdot r^{-'} < m^{-^*}$, where $r^{-'}$ is the conjugate exponent of r^-, i.e., $\frac{1}{r^-} + \frac{1}{r^{-'}} = 1$.

Using the Hölder inequality we obtain

$$\int_\Omega |V(x)| \cdot |u|^{m^-} dx \leq |V|_{r^-} \cdot |u|^{m^-}_{m^- \cdot r^{-'}}, \;\; \forall\, u \in W_0^{1,m^-}(\Omega). \tag{11}$$

Using inequalities (10) and (11) we obtain that inequality (9) holds.

Analogous we have that $r^- > \frac{N}{m^+}$, which implies that there exists a constant $C''_{V,\varepsilon}$ such that for all $u \in W_0^{1,m^+}(\Omega)$, the following relation:

$$\int_\Omega |V(x)| \cdot |u|^{m^+} dx \leq \varepsilon \int_\Omega |\nabla u|^{m^+} dx + C''_{V,\varepsilon} |V|_{r^-} \int_\Omega |u|^{m^+} dx \tag{12}$$

holds.

Taking account of the relation (7), $m^- \leq m^+ < p_1(x)$ for any $x \in \overline{\Omega}$ we have that $W \subset W_0^{1,m^\pm}(\Omega)$, therefore one can say that (9) and (12) hold for each $u \in W$.

Since the embedding $L^{r(x)}(\Omega) \hookrightarrow L^{r^-}(\Omega)$ is continuous we can have

$$\int_\Omega |V(x)| \cdot |u|^{m^-} dx \leq \varepsilon \int_\Omega |\nabla u|^{m^-} dx + C_{V,\varepsilon} |V(x)|_{r(x)} \int_\Omega |u|^{m^-} dx$$

and

$$\int_\Omega |V(x)| \cdot |u|^{m^+} dx \leq \varepsilon \int_\Omega |\nabla u|^{m^+} dx + C''_{V,\varepsilon} |V(x)|_{r(x)} \int_\Omega |u|^{m^+} dx.$$

Taking into account that $p_2(x) < m^- \leq m(x) \leq m^+ < p_1(x)$ for every $x \in \overline{\Omega}$ we point out that for all $u \in W$ we have:

$$\left| \int_\Omega \frac{V(x)}{m(x)} |u|^{m(x)} dx \right| \leq \frac{1}{m^-} \int_\Omega |V(x)| \cdot (|u|^{m^-} + |u|^{m^+}) dx.$$

Combining the last three inequalities from above we can conclude our proposition.

$\square$

Proposition 4.3. *Suppose that* (7) *and* $(HI_1)-(HI_4)$ *hold. Then one have:*

$$\int_\Omega \left(|\nabla u|^{m^-} + |\nabla u|^{m^+}\right) dx \leq \frac{2p_1^+}{c} \int_\Omega E_0(x, |\nabla u|)dx.$$

Proof. Using (HI_4) we obtain that:

$$\int_\Omega E_0(x, |\nabla u|)dx \geq \frac{1}{p_1^+} \int_\Omega [\mathcal{H}(x, |\nabla u|) + \mathcal{I}(x, |\nabla u|)] \cdot |\nabla u|^2 dx.$$

Furthermore using (HI_3) we have

$$\int_\Omega E_0(x, |\nabla u|)dx \geq \frac{c}{p_1^+} \int_\Omega |\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)} dx.$$

Now taking use of relation (7) and the previous inequality we obtain that:

$$\int_\Omega \left(|\nabla u|^{m^-} + |\nabla u|^{m^+}\right) dx \leq 2 \int_\Omega \left(|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}\right) dx,$$

hence

$$\int_\Omega \left(|\nabla u|^{m^-} + |\nabla u|^{m^+}\right) \leq \frac{2p_1^+}{c} \int_\Omega E_0(x, |\nabla u|)dx$$

which concludes our proof. □

Using the results obtained by Propositions 4.2 and 4.3 we can say that for every positive constant ε, there exists a positive constant C_ε such that:

$$\left| \int_\Omega \frac{V(x)}{m(x)} |u|^{m(x)} dx \right| \leq \varepsilon \int_\Omega E_0(x, |\nabla u|)dx + C_\varepsilon |V|_{r(x)} \int_\Omega \left(|u|^{m^-} + |u|^{m^+}\right) dx \tag{13}$$

for all $u \in W$.

The following lemma plays a crucial role in obtaining our further results.

Lemma 4.4. *The following relations hold:*

$$\lim_{\|u\| \to \infty} \frac{\mathcal{E}_V(u)}{\mathcal{F}(u)} = \infty \tag{14}$$

and

$$\lim_{\|u\| \to 0} \frac{\mathcal{E}_V(u)}{\mathcal{F}(u)} = \infty \tag{15}$$

Proof. Using (7) we can reveal that

$$q_1(x) \le m^- \le m^+ \le q_2(x), \quad \forall\, x \in \overline{\Omega}.$$

Therefore we can say that

$$|u(x)|^{m^-} + |u(x)|^{m^+} \le 2\left(|u(x)|^{q_1(x)} + |u(x)|^{q_2(x)}\right), \ \forall\, x \in \overline{\Omega},\ \forall\, u \in W.$$

Integrating the inequality over Ω we obtain that

$$\frac{\displaystyle\int_\Omega \left(|u|^{m^-} + |u|^{m^+}\right) dx}{\displaystyle\int_\Omega \left(|u|^{q_1(x)} + |u|^{q_2(x)}\right) dx} \le 2 \tag{16}$$

for all $u \in W$.

Let evaluate now the quantity:

$$\frac{\mathcal{E}_V(u)}{\mathcal{F}(u)} = \frac{E(u) + \displaystyle\int_\Omega \frac{V(x)}{m(x)} |u|^{m(x)} dx}{\displaystyle\int_\Omega \frac{1}{q_1(x)} |u|^{q_1(x)} dx + \int_\Omega \frac{1}{q_2(x)} |u|^{q_2(x)} dx}.$$

Using the relation (13) for any $\varepsilon \in (0,1)$ and taking account of (HI_3) and (HI_4) we obtain

$$\begin{aligned}
\frac{\mathcal{E}_V(u)}{\mathcal{F}(u)} \quad &\ge \quad \frac{\dfrac{(1-\varepsilon)c}{p_1^+} \displaystyle\int_\Omega \left(|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}\right) dx}{\dfrac{1}{q_2^-} \displaystyle\int_\Omega \left(|u|^{q_1(x)} + |u|^{q_2(x)}\right) dx} - \\
&- \quad \frac{C_\varepsilon |V(x)|_{r(x)} \displaystyle\int_\Omega \left(|u|^{m^-} + |u|^{m^+}\right) dx}{\dfrac{1}{q_2^-} \displaystyle\int_\Omega \left(|u|^{q_1(x)} + |u|^{q_2(x)}\right) dx}.
\end{aligned}$$

Using relations (7) and (16) we have that

$$\begin{aligned}
\frac{\mathcal{E}_V(u)}{\mathcal{F}(u)} \quad &\ge \quad \frac{\dfrac{(1-\varepsilon)c}{p_1^+} \displaystyle\int_\Omega \left(|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}\right) dx}{\dfrac{1}{q_2^-} \displaystyle\int_\Omega \left(|u|^{q_1(x)} + |u|^{q_2(x)}\right) dx} - \tilde{C}_1 |V(x)|_{r(x)} \\
&\ge \quad \frac{\dfrac{(1-\varepsilon)c}{p_1^+} \displaystyle\int_\Omega |\nabla u|^{p_1(x)} dx}{\dfrac{1}{q_2^-} \left(|u|_{q_1^-}^{q_1^-} + |u|_{q_1^+}^{q_1^+} + |u|_{q_2^-}^{q_2^-} + |u|_{q_2^+}^{q_2^+}\right)} - \tilde{C}_1 |V(x)|_{r(x)},
\end{aligned}$$

where $\tilde{C}_1$ is a positive constant.

By relation (7), we have the continuous embeddings $W \hookrightarrow L^{q_1^{\pm}}(\Omega)$ and $W \hookrightarrow L^{q_2^{\pm}}(\Omega)$, so we can find a positive constant $\tilde{C}_2$ such that

$$\frac{\mathcal{E}_V(u)}{\mathcal{F}(u)} \geq \frac{\dfrac{(1-\varepsilon)c}{p_1^+}\displaystyle\int_\Omega |\nabla u|^{p_1(x)}dx}{\tilde{C}_2\left(\|u\|^{q_1^-} + \|u\|^{q_1^+} + \|u\|^{q_2^-} + \|u\|^{q_2^+}\right)} - \tilde{C}_1|V(x)|_{r(x)}.$$

Since $\|u\| > 1$ in W we have that

$$\frac{\mathcal{E}_V(u)}{\mathcal{F}(u)} \geq \frac{\dfrac{(1-\varepsilon)c}{p_1^+}\|u\|^{p_1^-}}{\tilde{C}_2\left(\|u\|^{q_1^-} + \|u\|^{q_1^+} + \|u\|^{q_2^-} + \|u\|^{q_2^+}\right)} - \tilde{C}_1|V(x)|_{r(x)}.$$

for all $u \in W$, with $\|u\| > 1$.

Now, letting $\|u\| \to \infty$ and taking into account that $q_1^- \leq q_1^+ \leq q_2^- \leq q_2^+ < p_1^-$ we have that

$$\lim_{\|u\|\to\infty} \frac{\mathcal{E}_V(u)}{\mathcal{F}(u)} = \infty.$$

We proceed now to prove the second part of this lemma.

Using the fact that $p_2(x) < p_1(x)$ for all $x \in \overline{\Omega}$ we have the following continuous embedding

$$W \hookrightarrow W_0^{1,p_2(x)}(\Omega)$$

and so, if $\|u\| \to 0$, then $\|u\|_{p_2(x)} \to 0$.

Let us take for now on $\|u\| < 1$ and $\|u\|_{p_2(x)} < 1$. Moreover, using relations (7) and (8) we have the following continuous embedding $W_0^{1,p_2(x)}(\Omega) \hookrightarrow L^{q_1^{\pm}}(\Omega)$ which yields to the fact that there exist two positive constants $C_{q_1^+}$ and $C_{q_1^-}$ such that

$$\|u\|_{p_2(x)} \geq C_{q_1^+}|u|_{q_1^+} \quad \text{and} \quad \|u\|_{p_2(x)} \geq C_{q_1^-}|u|_{q_1^-}.$$

We note that we have the same behavior between the spaces $W_0^{1,p_2(x)}(\Omega)$ and $L^{q_2^{\pm}}(\Omega)$, so one have

$$\|u\|_{p_2(x)} \geq C_{q_2^+}|u|_{q_2^+} \quad \text{and} \quad \|u\|_{p_2(x)} \geq C_{q_2^-}|u|_{q_2^-}.$$

For every $u \in W$ with $\|u\| < 1$, we obtain similarly with the previous part that:

$$\begin{aligned}
\frac{\mathcal{E}_V(u)}{\mathcal{F}(u)} \quad & \geq \frac{\dfrac{(1-\varepsilon)c}{p_1^+}\displaystyle\int_\Omega |\nabla u|^{p_2(x)}dx}{\dfrac{1}{q_2^-}\left(|u|_{q_1^-}^{q_1^-} + |u|_{q_1^+}^{q_1^+} + |u|_{q_2^-}^{q_2^-} + |u|_{q_2^+}^{q_2^+}\right)} - \tilde{C}_1|V(x)|_{r(x)} \\
& \geq \frac{C_{p_1^+}\|u\|_{p_2(x)}^{p_2^+}}{\tilde{C}_3\left(\|u\|_{p_2(x)}^{q_1^-} + \|u\|_{p_2(x)}^{q_1^+} + \|u\|_{p_2(x)}^{q_2^-} + \|u\|_{p_2(x)}^{q_2^+}\right)} - \tilde{C}_1|V(x)|_{r(x)},
\end{aligned}$$

where $C_{p_1^+}, \tilde{C}_3 > 0$ are some constants.

Now using relation (7) and passing to the limit as $\|u\| \to 0$ (therefore, $\|u\|_{p_2(x)} \to 0$) we obtain that:

$$\lim_{\|u\|\to 0} \frac{\mathcal{E}_V(u)}{\mathcal{F}(u)} = \infty,$$

which concludes our proof.

□

Proof of Theorem 4.1.

We consider that $(u_n) \subset W \setminus \{0\}$ is a minimizing sequence for $\lambda_1(V)$, i.e.:

$$\lim_{n\to\infty} \frac{\mathcal{E}_V(u_n)}{\mathcal{F}(u_n)} = \lambda_1(V) \tag{17}$$

Taking account of the first relation in Lemma 4.4 we obtain that (u_n) is bounded in W. Using the property of reflexivity of the space W (passing eventually to a subsequence) we obtain that:

$$u_n \rightharpoonup u \quad \text{in } W.$$

By the compact embedding theorem for spaces with variable exponent, using the relation (7) we have that:

$$W \hookrightarrow L^{\theta(x)}(\Omega), \quad \text{where} \quad \theta(x) := \frac{m(x)\cdot r(x)}{r(x)-1},$$

$$W \hookrightarrow L^{q_1(x)}(\Omega)$$

and

$$W \hookrightarrow L^{q_2(x)}(\Omega).$$

Hence, we have that

$$u_n \to u \quad \text{in } L^{\theta(x)}(\Omega),$$
$$u_n \to u \quad \text{in } L^{q_1(x)}(\Omega)$$

and

$$u_n \to u \quad \text{in } L^{q_2(x)}(\Omega).$$

Using the Hölder inequality and relation (5) we obtain that

$$\lim_{n\to\infty} \mathcal{F}(u_n) = \mathcal{F}(u) \tag{18}$$

and

$$\lim_{n\to\infty} \int_\Omega V(x)|u_n|^{m(x)}dx = \int_\Omega V(x)|u|^{m(x)}dx. \tag{19}$$

Using relations (18) and (19) and the properties of $\mathcal{H}(\cdot,\cdot)$ and $\mathcal{I}(\cdot,\cdot)$ (for more details we refer to [12], Lemma 4.3) we have that:

$$E(u) \leq \liminf_{n\to\infty} E(u_n),$$

hence we obtain

$$\lambda_1(V) = \frac{\mathcal{E}_V(u)}{\mathcal{F}(u)}.$$

We only have left to prove that $u \not\equiv 0$.

Arguing by contradiction we assume that $u = 0$.

Therefore we have that $u_n \rightharpoonup 0$ in W and $u_n \to 0$ in $L^{h(x)}(\Omega)$, where $1 < h(x) < \frac{Np_1(x)}{N-p_1(x)}$ on $\overline{\Omega}$.

Using (18) and (19) we obtain that

$$\lim_{n\to\infty} (\mathcal{F}(u_n)) = 0 \tag{20}$$

and

$$\lim_{n\to\infty} \int_\Omega V(x)|u_n|^{m(x)}dx = 0. \tag{21}$$

Knowing that $\varphi \in (0, |\lambda_1(V)|)$ is fixed, by relation (17), for n large enough, we obtain

$$|\mathcal{E}_V(u_n) - \lambda_1(V)\mathcal{F}(u_n)| < \varphi\mathcal{F}(u_n)$$

or

$$(|\lambda_1(V)| - \varphi)\mathcal{F}(u_n) < \mathcal{E}_V(u_n) < (|\lambda_1(V)| + \varphi)\mathcal{F}(u_n).$$

Passing to the limit and using relation (20) we have that:

$$\lim_{n\to\infty} \mathcal{E}_V(u_n) = 0.$$

Combining the previous relation with relation (5) and by assumptions $(HI_1) - (HI_4)$ we obtain that $\|u_n\| \to 0$ in W as $n \to \infty$. Using that and relation (15) we obtain

$$\lim_{n\to\infty} \frac{\mathcal{E}_V(u_n)}{\mathcal{F}(u_n)} = \infty,$$

which is a contradiction. Thus $u \not\equiv 0$.

In order to complete the proof of our theorem we only have to show that u found before is a solution of the problem (P).

Firstly we remind that

$$\frac{\mathcal{E}_V(u)}{\mathcal{F}(u)} = \lambda_1(V) = \inf_{v\in W\setminus\{0\}} \frac{\mathcal{E}_V(v)}{\mathcal{F}(v)}.$$

We fix $v \in W \setminus \{0\}$ arbitrary and consider the application:

$$\omega \mapsto K(\omega) := \frac{\mathcal{E}_V(u+\omega v)}{\mathcal{F}(u+\omega v)}$$

which is defined in a neighborhood of the origin. It follows that $K'(0) = 0$, therefore

$$\left[\mathcal{E}_V'(u+\omega v)\mathcal{F}(u+\omega v) - \mathcal{E}_V(u+\omega v)\mathcal{F}'(u+\omega v)\right]\Big|_{\omega=0} = 0.$$

Hence,

$$\mathcal{F}(u)\left(\int_\Omega \left[\mathcal{H}(x,|\nabla u|) + \mathcal{I}(x,|\nabla u|)\right]\nabla u \nabla v dx + \int_\Omega V(x)|u|^{m(x)-2} uv dx\right) =$$

$$= \mathcal{E}_V(u)\int_\Omega \left(|u|^{q_1(x)-2} + |u|^{q_2(x)-2}\right) uv dx.$$

Using the definition of $\lambda_1(V)$ and the previous equality, we conclude that u solves (P) in weak sense, and so u is a weak solution of problem (P), which completes the proof of our theorem.

□

5 Concentration of the spectrum

In this section we will describe some spectral properties in relationship with the two Rayleigh-type quotients presented in the previous sections.

Theorem 5.1. *Let $\lambda \in \mathbb{R}$. Assume that $\lambda < \lambda_0(V)$. Then λ is not an eigenvalue for the problem (P).*

Proof. Arguing by contradiction we suppose that λ is an eigenvalue for the problem (P), which means that there exists $u_{\lambda_0} \in W \setminus \{0\}$ such that:

$$\int_\Omega [\mathcal{H}(x, |\nabla u_{\lambda_0}|) + \mathcal{I}(x, |\nabla u_{\lambda_0}|)] |\nabla u_{\lambda_0}||\nabla \varphi| dx + \int_\Omega V(x)|u_{\lambda_0}|^{m(x)-2} u_{\lambda_0} \varphi dx =$$

$$= \lambda \int_\Omega \left(|u_{\lambda_0}|^{q_1(x)-2} + |u_{\lambda_0}|^{q_2(x)-2} \right) u_{\lambda_0} \varphi dx,$$

for all $\varphi \in W$.

Taking $\varphi = u_{\lambda_0}$ we obtain that:

$$\int_\Omega [\mathcal{H}(x, |\nabla u_{\lambda_0}|) + \mathcal{I}(x, |\nabla u_{\lambda_0}|)] |\nabla u_{\lambda_0}|^2 dx + \int_\Omega V(x)|u_{\lambda_0}|^{m(x)} dx =$$

$$= \lambda \int_\Omega \left(|u_{\lambda_0}|^{q_1(x)} + |u_{\lambda_0}|^{q_2(x)} \right) dx.$$

So one have:

$$\begin{aligned} \lambda &= \frac{\int_\Omega [\mathcal{H}(x, |\nabla u_{\lambda_0}|) + \mathcal{I}(x, |\nabla u_{\lambda_0}|)] |\nabla u_{\lambda_0}|^2 dx + \int_\Omega V(x)|u_{\lambda_0}|^{m(x)} dx}{\int_\Omega \left(|u_{\lambda_0}|^{q_1(x)} + |u_{\lambda_0}|^{q_2(x)} \right) dx} \\ &\geq \inf_{\varphi \in W \setminus \{0\}} \frac{\int_\Omega [\mathcal{H}(x, |\nabla \varphi|) + \mathcal{I}(x, |\nabla \varphi|)] |\nabla \varphi|^2 dx + \int_\Omega V(x)|\varphi|^{m(x)} dx}{\int_\Omega \left(|\varphi|^{q_1(x)} + |\varphi|^{q_2(x)} \right) dx} \end{aligned}$$

for all $\varphi \in W$, which leads to a contradiction with the choice of λ. □

Finally we prove the last result studied in this paper.

Theorem 5.2. *For every $\lambda \in \mathbb{R}$, with $\lambda > \lambda_1(V)$, the problem (P) has a weak solution corresponding to its eigenvalue λ.*

Proof. Let us fix $\lambda \in (\lambda_1(V), \infty)$ and set

$$J_{V,\lambda} : W \to \mathbb{R}$$

such that

$$J_{V,\lambda}(u) := \mathcal{E}_V(u) - \lambda \mathcal{F}(u).$$

It is natural to say that $J_{V,\lambda} \in C^1(W, \mathbb{R})$ and its directional derivative is:

$$\langle J'_{V,\lambda}(u), \varphi \rangle = \langle \mathcal{E}'_V(u), \varphi \rangle - \lambda \langle \mathcal{F}'(u), \varphi \rangle,$$

for all $\varphi \in W$.

Therefore the nontrivial critical points of $J_{V,\lambda}$ are weak solutions for the problem (P), and every λ associated to these solutions represents an eigenvalue for (P).

Now, using hypotheses (HI_3), (HI_4) and taking $\varepsilon \in (0,1)$ in relation (13) we obtain that:

$$
\begin{aligned}
J_{V,\lambda}(u) \geq & \frac{(1-\varepsilon)c}{p_1^+} \int_\Omega \left(|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)} \right) dx - \\
& - C_\varepsilon |V(x)|_{r(x)} \int_\Omega \left(|u|^{m^-} + |u|^{m^+} \right) dx - \frac{\lambda}{q_2^-} \int_\Omega \left(|u|^{q_1(x)} + |u|^{q_2(x)} \right) dx.
\end{aligned}
$$

Using relation (16) and the fact that $\|u\| > 1$, there exists a constant $\tilde{C}_3 > 0$ such that:

$$
\begin{aligned}
J_{V,\lambda}(u) \quad & \geq \quad \frac{(1-\varepsilon)c}{p_1^+} \|u\|^{p_1^-} - \tilde{C}_3 \int_\Omega \left(|u|^{q_1(x)} + |u|^{q_2(x)} \right) dx \\
& \geq \quad \frac{(1-\varepsilon)c}{p_1^+} \|u\|^{p_1^-} - \tilde{C}_3 \left(|u|_{q_1(x)}^{q_1^-} + |u|_{q_1(x)}^{q_1^+} + |u|_{q_2(x)}^{q_2^-} + |u|_{q_2(x)}^{q_2^+} \right).
\end{aligned}
$$

Using the embedding results regarding variable exponent Lebesgue-Sobolev spaces, dictated by the relations (7) and (8) we have that there exists a constant $\tilde{C}_4 > 0$ such that:

$$
J_{V,\lambda}(u) \geq \frac{(1-\varepsilon)c}{p_1^+} \|u\|^{p_1^-} - \tilde{C}_4 \left(\|u\|^{q_1^-} + \|u\|^{q_1^+} + \|u\|^{q_2^-} + \|u\|^{q_2^+} \right).
$$

Taking $\|u\| \to \infty$ we obtain that

$$
\lim_{\|u\| \to \infty} J_{V,\lambda}(u) = +\infty,
$$

which means that our energy functional is coercive.

Now using the fact that $\mathcal{E}_V(u)$ is weakly lower semicontinuous (for more details we refer to [12]), and using arguments from the Chapter 3 of [19] and [9] $\mathcal{F}(u)$ is weakly-strongly continuous, we find that there exists a global minimum point $\eta \in W$ such that:

$$
J_{V,\lambda}(\eta) = \inf_{u \in W} J_{V,\lambda}(u).
$$

We prove in what follows that $\eta \neq 0$. Using the fact that $\lambda_1(V) < \lambda$, we may find $\varphi_\lambda \in W$ such that $\mathcal{E}_V(\varphi_\lambda) - \lambda \mathcal{F}(\varphi_\lambda) < 0$, hence

$$
J_{V,\lambda}(\varphi_\lambda) < 0.
$$

Since η is a global minimum point of $J_{V,\lambda}$, we obtain that $J_{V,\lambda}(\eta) < 0$, which yields that $\eta \neq 0$.

Therefore for every $\lambda \in (\lambda_1(V), \infty)$, the problem (P) has a weak solution η with its corresponding eigenvalue λ. $\square$

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